YouTube Arabic Logic video, Wilfrid Hodges.
This text is available at
`wilfridhodges.co.uk/arabic71.pdf`

The logical diagrams of al-Barakāt

Abū al-Barakāt al-Baghdādī

a Rabbi based in Baghdad
c. 1080–c. 1165

describes in Arabic a new method
for handling Aristotle’s logic of syllogisms

Aristotle introduced his **categorical syllogisms** in the 4th century BC.

His **categorical sentences** are the four sentences

Every A is a B.
No A is a B.
Some A is a B.
Some A is not a B.

or any of the sentences got from these by replacing A and B by *two distinct letters*, for example

Every B is an A.
No C is a W.
Some X is not a J.

We can think of the letters as standing for nonempty sets.

So for example

'Every A is a B'

means 'Everything in the set A is also in the set B'.

A **(categorical) syllogism** consists of three (categorical) sentences where

- the first has letters A and B in some order;
- the second has letters B and C in some order;
- the third has first letter A and second letter C, and has 'therefore' written before it;
- we can deduce the third sentence from the first two.

The first two sentences are called the **premises** and the third is called the **conclusion**. The first two sentences together are a **premise-pair**.

For example

Every A is a B.
Every B is a C.
Therefore every A is a C.

Another example:

Every B is an A.
Some B is not a C.
Therefore some A is not a C.

But consider the two premises

Some A is a B.
Some C is a B.

No conclusion follows!
(Remember the conclusion must be one of
'Every A is a C', No A is a C', 'Some A is a C' and
'Some A is not a C'.)

To express that there is no conclusion, we say that this premise-pair (i.e. pair of premises) is **sterile**.

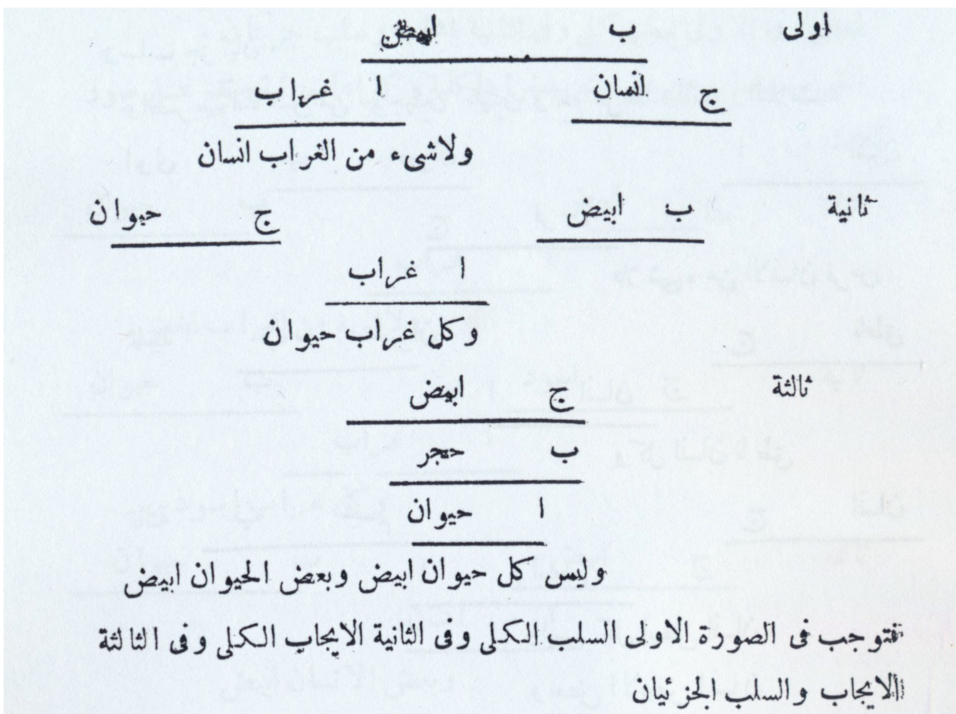
A premise-pair that does have a conclusion is said to be **productive**.

Al-Barakāt's method is a method for finding out,
given a premise-pair,
whether the premise-pair is sterile or productive;
and if it is productive, what its conclusion is.

Al-Barakāt’s main device is a **diagram** (Arabic *ṣūra*—today the word tends to mean photograph).

A (Barakāt) diagram is a collection of horizontal lines drawn on the page;
each line is labelled with a distinct letter.

Example from al-Barakāt manuscript;
three diagrams are shown:



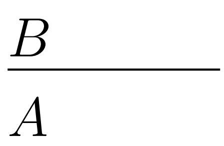
The diagrams indicate the relationship between the sets labelled by the letters, in terms of inclusion, overlap and non-overlap, as follows.

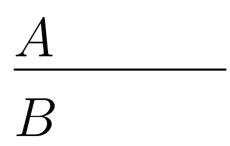
Like al-Barakāt, we begin with two-letter diagrams with the letters A and B standing for nonempty sets.

We say that such a diagram is a **model** of a sentence with letters A and B if the sentence is true when the sets are related as in the diagram.

Al-Barakāt points out that there are exactly five ways that A and B can be related.

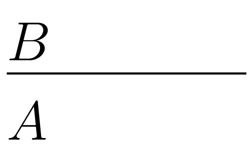
Case One: $A = B$. This is expressed by either of the two diagrams

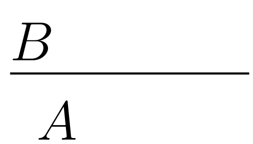


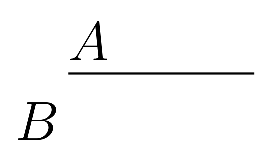


These are models of ‘Every A is a B’, ‘Some A is a B’, ‘Every B is an A’, ‘Some B is an A’.

Case Two: $A \subset B$, i.e. every member of A is a member of B, but some member of B is not in A. This is expressed by any of the diagrams





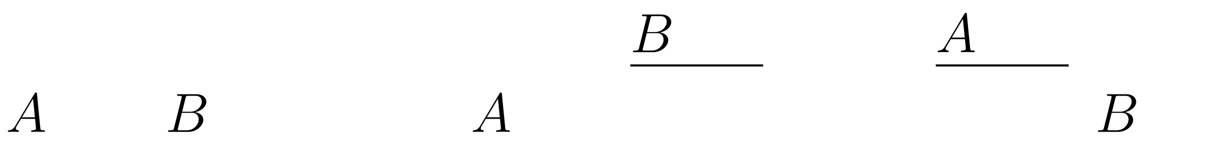


etc.

All these diagrams are models of ‘Every A is a B’, ‘Some A is a B’, ‘Some B is an A’, ‘Some B is not an A’.

Case Three: $B \subset A$. This is the same as Case Two, but with A and B the other way round,

Case Four: $A \perp B$, i.e. nothing is in both A and B. This is expressed by any of the diagrams



etc.

These are models of the four sentences 'No A is a B', 'Some A is not a B', 'No B is an A' and 'Some B is not an A'.

Case Five: A cuts across B, i.e. something is in both A and B, something is in A but not in B, and something is in B but not in A. Thus:

etc.

Al-Barakāt counts two diagrams that express the same case as equal.
Counting this way, for each pair of letters A, B there are exactly five diagrams.

Note. These five cases were rediscovered in 1816 by the French mathematician Joseph Gergonne.
Gergonne used circles instead of lines.
He introduced the symbol $\subset$ as in Case Two, to be short for the French word '$\subset$ontenue'.

Al-Barakāt uses three-letter diagrams (with three lines) to represent relationships between three nonempty sets A, B, C.
But for three letters the number of cases is not 5 but 109.

Al-Barakāt's criterion for productivity and conclusions:

> **Given a premise-pair (ϕ, ψ),**
> **list all the three-letter diagrams that are models of both ϕ and ψ.**
> **If there is a categorical sentence θ with first letter A and second letter C, such that all of the listed diagrams are models of θ,**
> **then (ϕ, ψ) is productive with conclusion θ.**

(If there is a choice of conclusions, choose the strongest.)

It can be shown that if (ϕ, ψ) is productive then it has at most sixteen models.
So if you find you have more than sixteen models, switch across to al-Barakāt's criterion for sterility, described below.

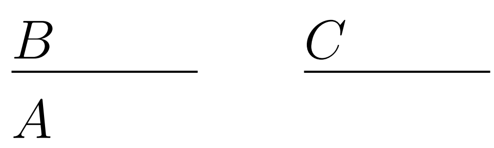
Example 1:

Every A is a B.
No B is a C.

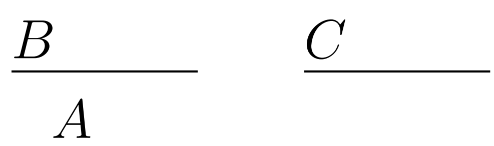
Workings

'No B is a C' has just one model:

We can expand this diagram to a three-letter diagram that it is a model of both premises.
We can do this in just two ways,
the first putting $A = B$:



and the second putting $A \subset B$:



Both diagrams are models of 'No A is a C'.
So the premise-pair is productive, and its conclusion is 'No A is a C'.
(Aristotle agreed but he gave no proof.)

Al-Barakāt's criterion for sterility:

Given a premise-pair (ϕ, ψ), look for

- **a three-letter model of ϕ and ψ that is also a model of 'Every A is a C';**
- **a three-letter model of ϕ and ψ that is also a model of 'No A is a C';**
- **a three-letter model of ϕ and ψ that is also a model of 'Some A is a C' and 'Some A is not a C'.**

If you can find all three, then the premise-pair is sterile.

It can be shown that if you can find the first two models, a model of the third kind can be found too. So in practice it suffices to find the first two models.

Example 2:

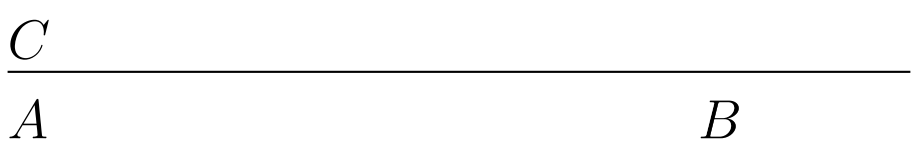
No B is an A.
Every B is a C.

Workings

Again ‘No B is an A’ has a unique model.

‘Every B is a C’ has two models:

We can combine the second of these models with the model of ‘No B is an A’ to get a model of ‘Every A is a C’:



and also to get a model of ‘No A is a C’:

So the premise-pair is sterile.

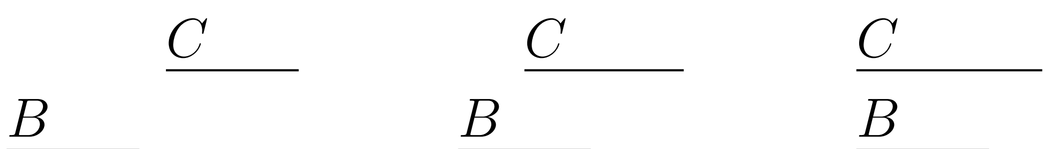
Example 3:

No A is a B.
Some C is not a B.

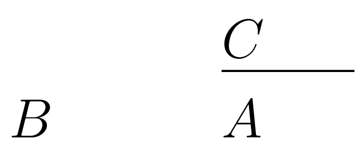
Heuristic. If answer is not clear at first, try proving sterility.

Workings

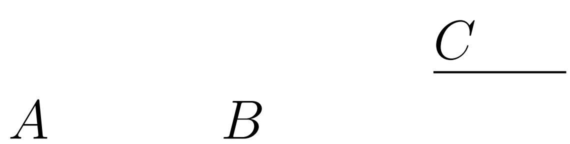
'Some C is not a B' has three models:



Looking to prove sterility, we can find a three-letter model of the premises and 'Every A is a C':



Likewise one for 'No A is a C':

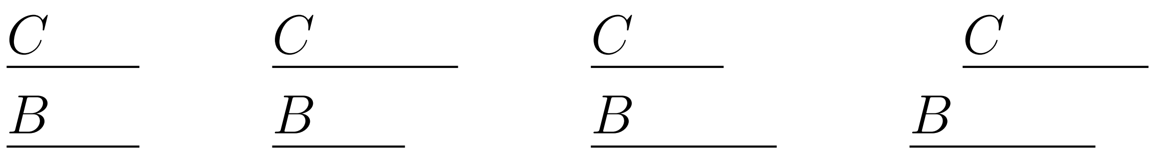


So the premise-pair is sterile.

Example 4:

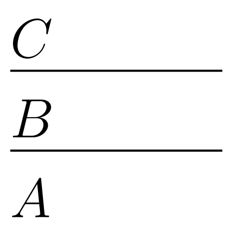
Every B is an A.
Some C is a B.

Workings. 'Some C is a B' has four models:



so this may be complicated. But we use the same heuristic and try to prove sterility.

We quickly find a three-letter model of the premises and 'Every A is a C':

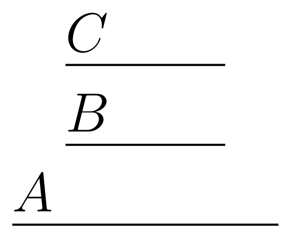


Looking for a model of 'No A is a C',
we see that a part of C is in B, and hence is also in A.
So every model of the premises is a model of
'Some C is an A', and hence also of 'Some A is a C'.

So 'Some A is a C' is a conclusion
and the premise-pair is productive.

Check: could the stronger sentence 'Every A is a C' also be a conclusion?

No, because the following model of the premises is not a model of 'Every A is a C':



So the premise-pair is productive with conclusion 'Some A is a C'.

Here are some more cases for you to try yourself as exercises.

Exercise 1

Every A is a B.
Every B is a C.

Exercise 2

No A is a B.
No C is a B.

Exercise 3

Some A is not a B.
Every C is a B.

Exercise 4

Every B is an A.
Every B is a C.

Exercise 5

Some A is a B.
No B is a C.

First remark

There are 109 relationships between three nonempty sets, but only 86 of them can be drawn with horizontal lines in al-Barakāt's manner. For example the following interpretation of the letters has no diagram:

- A The set of positive integers of the form $3m$ or $3m+1$.
- B The set of positive integers of the form $3m$ or $3m+1$.
- C The set of positive integers of the form $3m+1$ or $3m+2$.

Nevertheless al-Barakāt's method, using just the 86 line diagrams, always gives correct answers.
These facts are all proved in

Wilfrid Hodges, 'A correctness proof for al-Barakāt's logical diagrams', *Review of Symbolic Logic* (to appear, probably 2022).

Second remark

Al-Barakāt's diagrams are not an early version of Venn's diagrams, or of the diagrams used by Leibniz and Euler to support syllogisms.

The diagrams of Venn etc. are used to represent *sentences*,
so as to translate Aristotle's arguments into pictures.

By contrast the diagrams of al-Barakāt represent *models*, which is why his method should be counted as an early use of Tarski's 'model-theoretic consequence'.

Third remark

As we presented it, al-Barakāt's method is not purely mechanical.
We can easily make it mechanical by starting with a list of all 86 diagrams, and running through the list to check al-Barakāt's criteria for productivity and sterility.

But al-Barakāt himself used the diagrams more or less as we did above, as an aid to rational thinking about what models there are.

The medieval Arabic scholars seem not to have had the general idea of an algorithm.
They knew several important examples of algorithms, including the famous quadratic algorithm of al-Khwārizmī after whom algorithms are named.
But apparently they never conceptualised what the different algorithms have in common.

On this see
Wilfrid Hodges, 'Medieval Arabic notions of algorithm: some further raw evidence', in *Fields of Logic and Computation III, Essays Dedicated to Yuri Gurevich on the Occasion of his 80th Birthday*, ed. Andreas Blass et al., Lecture Notes in Computer Science 12180, Springer 2020, pp. 133-146.