Identifying Ibn Sīnā's hypothetical logic: II. Interpretations

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1 Introduction

This is one of three essays which jointly aim to describe the hypothetical logic of Ibn Sīnā's *Qiyās* vi.2 [22], referred to in [16] and [10] as Propositional Logic 3, PL3 for short. The paper [15] assembles the immediate evidence for the meanings of the sentence forms of this logic. It brings up to date the preliminary work that Nicholas Rescher published in 1964 [28], unfortunately before the detailed evidence on Ibn Sīnā's hypothetical logic in *Qiyās* was available. The monograph [17] uses the results of [15] to translate *Qiyās* vi.2 both into English and into modern logic. The logic is both sound and original; it contains a complete calculus for an extension of categorical logic allowing both affirmative and negative terms, and introduces some inference forms not recorded publicly in the West until De Morgan and Boole ([3] p. 140) in the nineteenth century. The strength and significance of this logical system provides reinforcing evidence for the correctness of the readings in [15]. But in this section of *Qiyās* Ibn Sīnā also makes an original contribution to methods of proving nonproductivity by means of interpretations. The analysis of this aspect of *Qiyās* vi.2 raises issues that need separate treatment, which is undertaken in the present paper.

On one reading of *Qiyās* vi.2, this text of Ibn Sīnā introduces, probably for the first time, tools of model theory which are standard in modern work, notably the notion of a universe of discourse. But this reading of Ibn Sīnā is based on what he does, not on what he says he is doing, and we need to examine his text closely to be sure we are not reading later ideas back into his work. The issues involved in this paper are more complex and obscure than those discussed in [15] and the hypothetical logic part of [17], and the conclusions reached are accordingly more tentative. It might be correct to say that what Ibn Sīnā achieved with his sterility proofs in *Qiyās* vi.2 is an Arabic parallel to David Hilbert's independence proofs in his *Grundlagen der Geometrie* [11] of 1899; but some doubts remain about how far Ibn Sīnā understood what he was doing.

2 Formulas of PL3

Ibn Sīnā's logic PL3, which he sets out in *Qiyās* vi.2, uses over forty sentence forms and a complex array of logical rules for translating between these forms. There are ten basic sentence forms as follows; for each we give a symbolic name, a translation into logical notation and a translation into English. Ibn Sīnā's own Arabic wordings for these forms, and the justifications for the modern formulas given as their translations, are discussed in [15].

symbolic name	logical formula	English translation
$(a, mt)(p, q)$	$\forall\tau\ (p\tau \to q\tau)$	Whenever p, q.
$(e, mt)(p, q)$	$\forall\tau\ (p\tau \to \neg q\tau)$	It is never the case when p that q.
$(i, mt)(p, q)$	$\exists\tau\ (p\tau \wedge q\tau)$	Sometimes p and q.
$(o, mt)(p, q)$	$\exists\tau\ (p\tau \wedge \neg q\tau)$	It is not always the case when p that q.
$(a, mn)(p, q)$	$\forall\tau\ (p\tau \vee q\tau)$	Always either p or q.
$pe, mn)_\alpha(p, q)$	$\forall\tau\ (p\tau \to q\tau)$	It is never the case that p, other than when q.
$(e, mn)_\beta(p, q)$	$\forall\tau\ (\neg p\tau \to \neg q\tau)$	It is never the case that q, other than when p.
$(i, mn)_\alpha(p, q)$	$\exists\tau\ (p\tau \wedge \neg q\tau)$	At some time other than when q, p.
$(i, mn)_\beta(p, q)$	$\exists\tau\ (\neg p\tau \wedge q\tau)$	At some time other than when p, q.
$(o, mt)(p, q)$	$\exists\tau(\neg p\tau \wedge \neg q\tau)$	It is not always the case that either p or q.

The quantifiers $\forall\tau$ and $\exists\tau$ are taken as ranging over times, and $p\tau$ is read as 'The sentence p is true at time τ' (though we will need to review this reading later). The two letters p and q can be replaced thoughout by any other pair of distinct letters. For example $(o, mt)(r, s)$ is a different formal sentence from $(o, mt)(p, q)$, but both sentences have the same sentence form.

Each of these ten sentence forms gives rise to three other sentence forms, got by adding a bar $^-$ above one or both of the letters p and q. The bar $^-$ is interpreted as negation. So for example $(a, mt)(\overline{p}, q)$ has the logical formula

(1) $$\forall\tau\ (\neg p\tau \to q\tau).$$

The letters p and q are called the 'term letters' of the sentences above, or for brevity simply the 'letters' of the sentences. The letters p, q, together with their bar if they have one, are known as the 'clauses' or 'terms' of the sentence. Thus in $(a, mt)(\overline{p}, q)$ the first term or clause is $\overline{p}$, and the first term letter is p; the second term letter is q, which is the same as the second clause.

This almost exhausts the sentence forms of PL3. A further two forms will be introduced at the end of this section.

A clause is called 'affirmative' if it has no bar, and 'negative' if it has a bar. Thus $(a, mt)(\overline{p}, q)$ has a negative first clause and an affirmative second clause. Sentences of the forms $(a, \ldots)$ or $(i, \ldots)$ are said to be 'affirmative', and sentences of the forms $(e, \ldots)$ or $(o, \ldots)$ are said to be 'negative'. Note that there is no connection between the sentence being affirmative and its clauses being affirmative; a negative sentence can have affirmative clauses and vice versa. For a sentence, the property of being affirmative or of being negative is called its 'quality'. For a clause, being affirmative or being negative is called its 'parity'.

Sentences of the forms $(a, \ldots)$ or $(e, \ldots)$ are said to be 'universal', and those of the forms $(i, \ldots)$ or $(o, \ldots)$ are said to be 'existential'. Being universal or being existential is the 'quantity' of a sentence.

The letters mt and mn are short for the Arabic names *muttaṣil* and *munfaṣil* for these two kinds of sentence. The original reasons for these names lose most of their justification in PL3, but some suggestion of a connection with logical meets and differences survives, and so we translate *muttaṣil* as 'meet-like' and *munfaṣil* as 'difference-like'. (The letters a, e, i and o in this context are a Scholastic shorthand; they were not known to Ibn Sīnā.)

Readers who compare this description of the sentence forms of PL3 with Ibn Sīnā's text as translated in Appendix A below will notice that he doesn't use single letters p or q to stand for sentences. Instead he uses schematic sentences like 'H is Z' or 'C is D', and his version of the bar is to write 'H is not Z' or 'C is not D'. In this he is following the practice of his predecessors. At other places in his hypothetical logic the separate letters C, D etc. play a role; see for example *Qiyās* vi.4 and vi.5, and the historical discussion by Maróth [25] pp. 164–168. But in *Qiyās* vi.2 Ibn Sīnā treats 'H is Z' as an unanalysed unit, which could stand for sentences that are not of the form 'H is Z' anyway (such as 'There is (not) a vacuum' in Hyp29 or Hyp36 in Appendix A). So for purposes of studying PL3 we will use the single letters p, q rather than Ibn Sīnā's more complex expressions.

There are two cases where Ibn Sīnā allows sentences of PL3 to be given stronger meanings. These have to be considered as a further two sentence forms, though Ibn Sīnā writes them in Arabic in the same way as when they are read with their original weaker meanings. The first case is that the sentence $(a, mt)(p, q)$ is sometimes read as

(2) $$(\forall\tau(p\tau \to q\tau) \land \exists\tau p\tau).$$

The added implication, that the sentence p is sometimes true, is known as 'existential import'. The second case is that a *munfaṣil* sentence that normally expresses

(3) $$\forall\tau(\neg p\tau \to q\tau)$$

is sometimes read as expressing

(4) $$\forall\tau(\neg p\tau \leftrightarrow q\tau).$$

With this stronger meaning the sentence is described as 'strict' (*ḥaqīqī*). In the sterility proofs that will concern us in this paper, Ibn Sīnā never makes any use of a premise-pair containing a premise that is read as either strict or having existential import. So for the purposes of this paper we lose nothing by ignoring these stronger readings of sentences, and henceforth we will ignore them. In Section 12 below we will need to examine the effects of making an assumption of the form $\exists\tau p\tau$, but there will be no need for us to think of this as part of the meanings of the premises being studied.

3 Equivalences and the AM fragment

Since we have given modern logical formulas for all the formal sentences of PL3, we can apply notions of modern logic to these sentences. For example sentences ϕ and ψ 'logically entail' sentence θ, in symbols

(5) $$\phi, \psi \vdash \theta,$$

if the formula for θ is derivable from the formulas for ϕ and ψ in first-order logic. We also read (5) as 'θ is a logical consequence of ϕ and ψ'. We say that ϕ and ψ are 'logically equivalent' if the formula for ϕ logically entails that for ψ and vice versa.

For each sentence ϕ of PL3 there is at least one sentence ψ of PL3 that is logically equivalent to the negation of ϕ. We describe any such sentence ψ as a (or the) 'contradictory negation' of ϕ.

We will need to use four kinds of transformation between sentences of PL3, namely parity switch, relettering, generous relettering and conversion.

By a 'parity switch' on a set L of letters, we mean a transformation σ of the set of barred and unbarred letters, that for each letter p in L switches p to $\overline{p}$ and $\overline{p}$ to p. Then σ can be used as a transformation of PL3 sentences ϕ, replacing each clause t by the clause σt; we write $\sigma\phi$ for the resulting sentence. This transformation σ doesn't take sentences to logically equivalent sentences, and it doesn't preserve the parity of clauses. But it does preserve the quantity and quality of sentences, and the first and second term letters of the sentence $\sigma\phi$ are the same as those of the sentence ϕ. Also, and most important, if θ is a logical consequence of ϕ and ψ, then $\sigma\theta$ is a logical consequence of $\sigma\phi$ and $\sigma\psi$. In other words, parity switches preserve logical consequence.

By a 'relettering' we mean a permutation π of the set of letters used as term letters, which takes each letter p to a letter πp. Then π can be used as a transformation of PL3 sentences, by replacing each term letter p of a sentence ϕ by πp; we call the resulting sentence $\pi\phi$. For example if ϕ is $(a, mt)(\overline{r}, q)$, πr is s and πq is r, then $\pi\phi$ is $(a, mt)(\overline{s}, r)$. Reletterings π in general don't take sentences to logically equivalent sentences, and unlike parity switches they alter the term letters of a sentence. But like parity switches, they preserve logical consequence. Also $\pi\phi$ has the same quantity and quality as ϕ, and its first and second terms have the same parities as those of ϕ.

By a 'generous relettering' we mean a transformation of sentences that consists of a relettering followed by a parity switch. The relettering can be the identity permutation and the permutation switch can be on the empty set of letters; so every relettering is a generous relettering, and likewise every parity switch is a generous relettering. Generous lettering preserves logical consequence. If ϕ and ψ are logically equivalent, then applying the same generous relettering to ϕ and to ψ gives two sentences that again are logically equivalent.

The fourth kind of transformation of PL3 sentences is 'conversion', which converts each sentence to a logically equivalent sentence of a similar form but with the two term letters in the opposite order. Conversions for meet-like sentences are

(6)
$$\begin{array}{lll} (a, mt)(p, q) & \text{converts to} & (a, mt)(\overline{q}, \overline{p}); \\ (e, mt)(p, q) & \text{converts to} & (e, mt)(q, p); \\ (i, mt)(p, q) & \text{converts to} & (i, mt)(q, p); \\ (o, mt)(p, q) & \text{converts to} & (o, mt)(\overline{q}, \overline{p}). \end{array}$$

For example $(a, mt)(r, \overline{q})$ converts to $(a, mt)(q, \overline{r})$. Conversions for the difference-like sentences are left to the reader.

If you convert a sentence and then convert again, you get back to the original sentence. Every PL3 sentence converts to another PL3 sentence; and since the process is invertible, every PL3 sentence is the result of converting another PL3 sentence. (This would be false if we included existential import.)

Every sentence of PL3 (still ignoring existential import and strictness) is logically equivalent to an affirmative meet-like PL3 sentence with the term letters in the same order. This fact allows us to boil down the whole of Ibn Sīnā's discussion of sterility to a fragment of PL3 with just eight sentence forms, namely the affirmative meet-like forms

(7) $$\begin{array}{l}(i,mt)(p,q),\ (i,mt)(p,\overline{q}),\ (i,mt)(\overline{p},q),\ (i,mt)(\overline{p},\overline{q}),\\ (a,mt)(p,q),\ (a,mt)(p,\overline{q}),\ (a,mt)(\overline{p},q),\ (a,mt)(\overline{p},\overline{q}).\end{array}$$

We call sentences of these eight forms 'AM sentences', and they form the 'AM fragment' of PL3.

Note that each of the four AM forms in the upper row has a contradictory negation in the lower row (and of course vice versa). For example $(a,mt)(p,\overline{q})$ is the contradictory negation of $(i,mt)(p,q)$.

4 Premise-pairs, productive and sterile

In PL3 a 'premise-pair' is an ordered pair of sentences of PL3 with the properties:

(a) The two sentences have exactly one term letter in common; this is called the 'shared letter' (*mushtarak*).

(b) The shared letter is not both in the first clause of the first premise and in the second clause of the second premise.

(Condition (b) expresses Ibn Sīnā's rejection of the fourth figure of the syllogism; see Rescher [29].) By a 'candidate conclusion', or more briefly a 'candidate', for this premise-pair we mean an AM sentence whose first term letter is the non-shared letter in the first premise, and whose second term letter is the non-shared letter in the second premise.

For example here is a premise-pair:

(8) $$(a,mn)(r,q),\ (i,mt)(p,\overline{q}).$$

It has eight candidates, namely the eight sentences

(9) $$\begin{array}{l}(i,mt)(r,p),\ (i,mt)(r,\overline{p}),\ (i,mt)(\overline{r},p),\ (i,mt)(\overline{r},\overline{p}),\\ (a,mt)(r,p),\ (a,mt)(r,\overline{p}),\ (a,mt)(\overline{r},p),\ (a,mt)(\overline{r},\overline{p}).\end{array}$$

If ϕ, ψ are a premise-pair, a 'conclusion' of the premise-pair is a candidate for the premise-pair which is also a logical consequence of the premise-pair. A premise-pair is 'productive' if it has a conclusion, and 'sterile' or 'nonproductive' if it has no conclusion. (The term 'sterile', *ᶜaqīm*, seems to have been introduced by Ibn Sīnā, and reflects a common metaphor of Arabic logic, that the two premises of an inference are parents of the conclusion. See for example the Brethren of Purity, [2] xiii.1, p. 114.)

Ibn Sīnā's *Qiyās* vi.2 is a study of productive and sterile premise-pairs in PL3. He gives a tightly constructed list of 108 premise-pairs. In principle he tells us, for each premise-pair listed, whether it is productive or sterile; if it is productive, he states and proves a conclusion of it. In practice he ignores many cases and gives only cursory treatment to others. Roughly sixty premise-pairs get full treatment; of these, roughly forty are productive and the remainder are sterile. He is nearly always right about which are productive. There are just two cases (Hyp9 and Hyp18) which he incorrectly pronounces sterile, and one case (Hyp71) that he incorrectly thinks is productive. For Hyp71 he offers a faulty proof, but he gives no supporting arguments for his view of Hyp9 and Hyp18.

He lists the 108 premise-pairs in twelve groups of nine. Four of these groups (numbered A1, B1, C1 and D1 in [17]) list between them thirty-six distinct premise-pairs. Group A2 repeats the premise-pairs in A1 but subject to a parity switch (the same one throughout A2); then group A3 lists the same premise-pairs again but with a different parity switch. The same applies to the groups B1, B2, B3, and similarly with C and D. We will describe two premise-pairs as being 'parity variants' of each other if one comes from the other by a parity switch. Thus the 108 premise-pairs can be regrouped into triplets, where all the premise-pairs within a triplet are parity variants of each other.

For each premise-pair that he discusses, we will refer to the text discussing it as an 'item'. In [17] the items are numbered Hyp1, Hyp2 etc., and we follow that numbering here. A typical item describes a premise-pair and then comments on its conclusion or its sterility. The premise-pair will be stated in Arabic, and unfortunately Ibn Sīnā sometimes uses the same Arabic wording to express logically different meanings; for this reason we have to distinguish the 'verbal' premise-pairs that he lists from the 'logical' premise-pairs that are expressed by our logical formulas.

Ibn Sīnā gives full details of sterility proofs for sixteen premise-pairs. Between them they contain representatives of eight triplets. Sometimes he writes out full proofs for all three premise-pairs in a triplet, and sometimes he treats only one or two of the premise-pairs in a triplet.

Below we list the sixteen premise-pairs for which Ibn Sīnā gives full sterility proofs, together with all the other premise-pairs in their triplets. We make some adjustments to ease comparisons. Thus Ibn Sīnā's premise-pairs contain sentences of all the forms allowed in PL3, but we bring all the premises to equivalent sentences of the AM fragment. Also we use q for the shared letter, r for the letter occurring only in the first premise and p for the letter occurring only in the third; Ibn Sīnā's own lettering for the clauses is not so uniform.

The eight resulting triplets are as follows, together with the numbering of the items in which Ibn Sīnā gives sterility proofs. (See Appendix A below for translations of the texts of these items.) The first three triplets are taken together because all nine premise-pairs are parity variants.

	item	first premise	second premise
Triplet 1	Hyp3	$(a, mt)(r, q)$	$(i, mt)(q, \overline{p})$
	Hyp12	$(a, mt)(r, q)$	$(i, mt)(q, p)$
		$(a, mt)(r, \overline{q})$	$(i, mt)(\overline{q}, \overline{p})$
Triplet 2		$(a, mt)(\overline{r}, q)$	$(i, mt)(q, p)$
	Hyp52	$(a, mt)(r, q)$	$(i, mt)(q, p)$
	Hyp59	$(a, mt)(\overline{r}, \overline{q})$	$(i, mt)(\overline{q}, p)$
Triplet 3		$(a, mt)(\overline{r}, q)$	$(i, mt)(q, \overline{p})$
		$(a, mt)(r, q)$	$(i, mt)(q, \overline{p})$
	Hyp63	$(a, mt)(\overline{r}, \overline{q})$	$(i, mt)(\overline{q}, \overline{p})$

The nine premise-pairs in the next three triplets are also parity variants of each other:

	item	first premise	second premise
Triplet 4	Hyp7	$(a, mt)(r, q)$	$(a, mt)(\overline{q}, \overline{p})$
	Hyp16	$(a, mt)(r, q)$	$(a, mt)(\overline{q}, p)$
	Hyp22	$(a, mt)(r, \overline{q})$	$(a, mt)(q, \overline{p})$
Triplet 5		$(a, mt)(r, q)$	$(a, mt)(\overline{q}, p)$
	Hyp10	$(a, mt)(r, q)$	$(a, mt)(\overline{q}, \overline{p})$
	Hyp19	$(a, mt)(r, \overline{q})$	$(a, mt)(q, p)$
Triplet 6	Hyp47	$(a, mt)(\overline{r}, \overline{q})$	$(a, mt)(q, p)$
	Hyp53	$(a, mt)(r, \overline{q})$	$(a, mt)(q, p)$
	Hyp60	$(a, mt)(\overline{r}, q)$	$(a, mt)(\overline{q}, p)$

Two triplets remain:

	item	first premise	second premise
Triplet 7	Hyp29	$(a, mt)(q, r)$	$(a, mt)(q, p)$
	Hyp36	$(a, mt)(q, r)$	$(a, mt)(q, \overline{p})$
		$(a, mt)(\overline{q}, r)$	$(a, mt)(\overline{q}, p)$

	item	first premise	second premise
Triplet 8	Hyp67	$(a, mt)(r, q)$	$(a, mt)(p, q)$
		$(a, mt)(\overline{r}, q)$	$(a, mt)(p, q)$
		$(a, mt)(r, \overline{q})$	$(a, mt)(p, \overline{q})$

The fact that the premise-pairs within a triplet are parity variants of each other is built into Ibn Sīnā's organisation of *Qiyās* vi.2, and he was obviously well aware of it. In fact he refers to it explicitly in Hyp22, Hyp43 and Hyp56. On the other hand the fact that premise-pairs in two different triplets are parity variants is partly an artefact of our reduction of all premise-pairs to the AM fragment of PL3. In Ibn Sīnā's text the different triplets use different sentence forms of PL3. We have no direct evidence that he realised this close relationship between different triplets; he never refers to it explicitly.

There is a theorem that goes a long way towards explaining the lack of variation between Ibn Sīnā's examples.

Theorem 1 *Every sterile premise-pair of AM sentences, if it contains at least one universal sentence, can be brought by a combination of generous relettering, conversion and permutation of the order of the premises (all of them operations that preserve sterility) to one of the following two forms:*

$$(a, mt)(r, q),\ (i, mt)(q, p).$$
$$(a, mt)(r, q),\ (a, mt)(\overline{q}, p).$$

Proof sketch. (See [16] for a full proof.) By permuting the premises if necessary, we can ensure that the first premise is universal. By converting each premise if necessary, we can ensure that the shared term letter is second in the first premise and first in the second premise. A generous relettering then makes r the first clause of the first premise, q the second clause of the first premise and p the second clause of the second premise. This leaves open whether the first clause of the second premise is q or $\overline{q}$. We inspect separately the case where the second premise is existential and the case where it is universal. It can be seen in each of these cases that it is necessary and sufficient for sterility that this clause has the form stated in the theorem. □

We refer to sterile premise-pairs of the first kind in the theorem, i.e. sterile premise-pairs consisting of one universal and one existential sentence, as of 'type ∀∃'. Sterile premise-pairs of the second kind, i.e. those consisting of two universal sentences, are of 'type ∀∀'. If we considered also premise-pairs where both premises are existential, we would have to add just two further premise-pairs to the list in the theorem. But Ibn Sīnā's scheme for listing premise-pairs excludes premise-pairs with both premises existential.

There is an easy test for sterility in the AM fragment. The letters of an AM sentence are described as 'distributed' or 'undistributed' in the sentence according to the following chart:

p distributed?	q distributed?	sentence
yes	yes	$(i, mt)(\overline{p}, \overline{q})$, $(a, mt)(p, \overline{q})$
yes	no	$(i, mt)(\overline{p}, q)$, $(a, mt)(p, q)$
no	yes	$(i, mt)(p, \overline{q})$, $(a, mt)(\overline{p}, \overline{q})$
no	no	$(i, mt)(p, q)$, $(a, mt)(\overline{p}, q)$

The following test is a special case of a more general test proved in [16].

Theorem 2 *Let ϕ, ψ be a premise-pair in the AM fragment, with shared letter q. Then ϕ, ψ is sterile if and only if q is either distributed in both ϕ and ψ, or undistributed in both ϕ and ψ.* □

It's clear that Ibn Sīnā was completely unaware of this test, since it would have saved him several errors and no end of effort.

5 Aristotle's method

The rest of this paper is devoted to Ibn Sīnā's proofs of the sterility of the premise-pairs listed above. He has just one method, and it clearly derives from the method that Aristotle used in the *Prior Analytics* to prove the non-productivity of some premise-pairs. The main tool is what in modern logic would be called an interpretation. Let me define this notion for PL3; the notion can be adapted straightforwardly to other logics.

By an 'interpretation' for PL3 we mean a list I of term letters, which assigns to each term letter p a natural language sentence $p[I]$. We define

$\overline{p}[I]$ to be the negation of the sentence $p[I]$. We call $p[I]$ the 'reading' of p in I; likewise $\overline{p}[I]$ is the 'reading' of $\overline{p}$ in I.

If ϕ is a sentence of PL3 and I is an interpretation whose list of letters includes the term letters of ϕ, we write $\phi[I]$ for the sentence got by writing $t[I]$ in place of each clause t of ϕ. We say that I 'verifies' ϕ, or that it is a 'model' of ϕ, if the sentence $\phi[I]$ is true; if $\phi[I]$ is false we say that I 'falsifies' ϕ. We say that I is a 'model' of a collection of sentences if it is a model of each sentence in the collection.

For example, using our notation, in Hyp19 Ibn Sīnā considers the sentence $(a, mt)(r, \overline{q})$, i.e. 'Whenever r, $\overline{q}$ '. He gives an interpretation I such that $r[I]$ is 'This is a human' and $q[I]$ is 'This is a stone'. So $\overline{q}[I]$ is 'This is not a stone', and $((a, mt)(r, \overline{q}))[I]$ is the sentence 'Whenever this is a human, this is not a stone'. The sentence is presumably understood as true, so that I is a model of $(a, mt)(r, \overline{q})$.

In his *Prior Analytics* Aristotle regularly applies the same method for proving nonproductivity of premise-pairs, both in categorical and in modal logic. He never explains the method, but it can be read off from his examples. In describing the method I will assume some knowledge of Aristotle's categorical syllogistic.

Aristotle's categorical syllogistic has four sentence forms, and the logical formulas below describe how Ibn Sīnā understood these forms:

$(a)(A, B)$:	Every A is a B.	$(\forall x\ (Ax \to Bx) \wedge \exists x Ax)$
$(e)(A, B)$:	No A is a B.	$\forall x\ (Ax \to \neg Bx)$
$(i)(A, B)$:	Some A is a B.	$\exists x\ (Ax \wedge Bx)$
$(o)(A, B)$:	Not every A is a B.	$(\exists x\ (Ax \wedge \neg Bx) \vee \forall x \neg Ax)$

The letters A, B are the 'terms' of the given sentences. An interpretation for this logic will assign common nouns or common noun phrases (instead of sentences) to the term letters A, B etc.

Aristotle's method for categorical logic is as follows, given premises ϕ and ψ with minor term C, middle (= shared) term B and major term A. Suppose the premise-pair ϕ, ψ is nonproductive. Aristotle proves the nonproductivity by giving two interpretations I, J, each of which lists the terms C, B, A. Aristotle's method requires that

R1: Both the interpretations I and J are models of both premises.

R2: The sentence 'Every C is an A' is verified by I, and the sentence 'No C is an A' is verified by J.

R3: I and J agree in the nouns or phrases that they assign to C, and they agree in the nouns or phrases that they assign to B. (There is no such requirement on A.)

The candidate conclusions for these two premises are the four sentences

(10) $$(a)(C, A),\ (e)(C, A),\ (i)(C, A),\ (o)(C, A).$$

So Aristotle's method is justified if we can show that R1–R3 imply that none of the four candidates (10) is a logical consequence of ϕ and ψ.

Our usual understanding of Aristotle's method rests on the following assumption:

(11) If ϕ and ψ logically entail θ then every interpretation that is a model of ϕ and ψ is also a model of θ.

Because of an obvious resemblance to ideas in Tarski [31] we will call this assumption 'Tarski's principle'. In the surviving literature it seems that Tarski's principle in this form was first stated by Alexander of Aphrodisias [1] 238.35f, who observed that it needed justification.

Assuming R1 and R2, we show as follows that $(a)(C, A)$ is not a conclusion of ϕ and ψ. If it was a conclusion of them, then it would be a logical consequence of them, so that by R1 and Tarski's principle, J would verify it. But by R2, J verifies the sentence $(e)(C, A)$, and $(e)(C, A)$ is incompatible with $(a)(C, A)$ in the sense that no interpretation can be a model of both these sentences. So J is not a model of $(a)(C, A)$; contradiction. The same argument proves that $(i)(C, A)$ is not a logical consequence of the premises. To prove that neither of the two negative sentences $(e)(C, A)$ and $(o)(C, A)$ is a logical consequence of the premises, we follow the same argument but with I in place of J. (We remark that R3 played no role in this argument.)

Aristotle uses the same method to prove nonproductivity in modal logic, but instead of $(a)(C, A)$ and $(e)(C, A)$ he uses the sentences 'Necessarily every C is an A' and 'Necessarily no C is an A'.

The natural language sentences $((a)(C, A))[I]$ and $((e)(C, A))[J]$ play a central role in Aristotle's method; clause R2 says precisely that both these sentences are true. The sentences can be described as the strongest sentences of the form $\theta[I]$ or $\theta[J]$ that are true, where θ is a candidate for ϕ and ψ. In the one place where Aristotle gives a name to these sentences, he calls them 'conclusion' (*sumpérasma*), and this terminology became standard at least up until the time of Ibn Sīnā. Thus in Hyp12 and Hyp16 Ibn Sīnā describes these sentences as 'yielded by' the matters or as 'following from' the terms; he uses the same terminology 'yields' and 'follows from' for logical consequences of premises.

The three papers [12], [13], [14] review the use of this terminology from Aristotle onwards. One standard feature of the terminology was that the true sentences were described as conclusions 'from the terms', not from the premises. Some authors (for example Paul the Persian) described genuine logical conclusions as 'necessary conclusions' or 'sound conclusions' to distinguish them from this other kind of conclusion.

To a modern ear the use of 'conclusion' for Aristotle's true sentences is disturbing, since there is no claim that they are logical conclusions from any premises. I will use 'pseudoconclusion' (of an interpretation I) to mean either a candidate sentence θ such that $\theta[I]$ is true, or the true sentence $\theta[I]$ itself. If one wants a name for Aristotle's method of proving nonproductivity, it seems reasonable to call it the method of pseudoconclusions. (Aristotle himself usually introduces any use of the method with the curt announcement 'terms'.)

Another useful piece of terminology was popularised by Alexander of Aphrodisias. In Aristotle's method the sentence $(e)(C, A)$ is used to show that neither $(a)(C, A)$ nor $(i)(C, A)$ is a conclusion from the given premises. Alexander expresses this by saying that the pseudoconclusion $(e)(C, A)$ 'rules out' (*anaireî*) the sentences $(a)(C, A)$ and $(i)(C, A)$. So Aristotle's method involves finding two pseudoconclusions which between them rule out all four candidates.

6 The failure of Aristotle's method in PL3

At least at first sight, all of Ibn Sīnā's proofs of sterility in PL3 are applications of Aristotle's method with minimal alterations. The letters in PL3 stand for sentences rather than nouns, so an interpretation is now a listing of letters that assigns sentences to the letters. For each sterile premise-pair, Ibn Sīnā provides two interpretations. Mostly he gives the interpretations without commentary, just as Aristotle did. But in a few cases he mentions the pseudoconclusions. Thus in Hyp3 he says 'In the first case the affirmative universal [sentence] is true, and in the second case the negative universal [sentence] is true', and in Hyp7 he says 'The one case makes an affirmative universal [sentence] true, and the other case makes a negative universal [sentence] true'. At Hyp29 he spells out the affirmative pseudoconclusion as 'what is true', and states that 'what is true' with the other interpretation is 'a negative proposition'.

These negative pseudoconclusions are not in the AM fragment, since AM sentences are by definition affirmative. But PL3 does have a sentence form that corresponds to Aristotle's $(e)(C,A)$, namely the negative sentence form

(12) $$(e,mt)(r,p),\ \text{formalised as}\ \forall\tau\ (r\tau \to \neg p\tau).$$

The logical symbolisation for this sentence is identical to that for the AM sentence $(a,mt)(r,\overline{p})$, showing that the two sentences are logically equivalent. Ibn Sīnā is well aware of this equivalence; he invokes it many times. So we can read all of Ibn Sīnā's sterility proofs as being carried out entirely within the AM fragment, using pseudoconclusions of the forms $(a,mt)(r,p)$ and $(a,mt)(r,\overline{p})$.

Ibn Sīnā always gives two interpretations, just as Aristotle did. In all but three of the arguments set out in *Qiyās* vi.2, the two interpretations give the same reading to r and the same reading to q, just as with clause R3 in Aristotle's method. One exception is Hyp3, where the readings given by the two interpretations are completely different. The other exceptions are the parity variants Hyp47 and Hyp53, where the two interpretations agree on q and p but differ on r—so that clause R3 holds back to front, so to say.

In short and with the three exceptions just mentioned, Ibn Sīnā writes exactly as if he was using Aristotle's method to prove sterility. He makes no remarks that suggest he has any other method in mind.

Unfortunately Aristotle's method doesn't work for PL3. The problem is that while premise-pairs in categorical logic have the four candidate conclusions (10), premise-pairs in PL3 have eight, even ignoring existential import and strictness, namely the eight sentences in (9). So each of the two pseudoconclusions must rule out at least four of these candidates. But without existential import or strictness, no sentence of PL3 rules out more than one candidate. Sentences with existential import or strictness will rule out two of the eight, but the arithmetic is still fatal for Aristotle's method.

There are no remarks anywhere in Ibn Sīnā's text to suggest that Aristotle's method might not be adequate for PL3. But perhaps not much can be inferred from this, because Ibn Sīnā says nothing to justify Aristotle's method either.

Ibn Sīnā strikes one as a writer who (to misquote the Mock Turtle) never went anywhere without a purpose. We can be sure that he had in mind *some* method for proving sterility. The question is what that method was. I know of only two plausible suggestions. Anybody who can think of a significantly different strategy that Ibn Sīnā might have been pursuing is welcome to rewrite the rest of this paper in the light of that strategy.

The first suggested method is Aristotle's method of pseudoconclusions.

The second suggested method is the model-theoretic method for proving nonentailment. It was introduced into modern logic through work of Peano and Hilbert in the 1890s, and was underpinned by Tarski's work in the mid 20th century on the foundations of model theory. In this method, suppose we are given a premise-pair ϕ and ψ, and the candidate conclusions are $\theta_1, \ldots, \theta_8$. Then we are required to produce a set of interpretations $I_1, \ldots, I_8$ such that

R1′: All the interpretations $I_1, \ldots, I_8$ are models of both premises.

R2′: Each sentence θ_i ($1 \leqslant i \leqslant 8$) is falsified by I_i.

So each candidate θ_i is ruled out, not by a pseudoconclusion but by an interpretation. There is no requirement that $I_1, \ldots, I_8$ must all be distinct; but the number of distinct interpretations must be at least two, since for example no interpretation can falsify both $(a, mt)(r, p)$ and its contradictory negation $(i, mt)(r, \overline{p})$. I can see no obvious a priori reason why two interpretations should be enough; but Ibn Sīnā always gives just two.

To try to settle the issue between these two suggested methods, we will run through Ibn Sīnā's sixteen sterility proofs, which contain thirty-two interpretations between them. We will assess how well they meet the requirements R1 and R2 of Aristotle's method on the one hand, and the requirements R1′ and R2′ of the model-theoretic method on the other hand. Since R1 and R1′ coincide, this boils down to three requirements.

The requirement R1 won't directly help to distinguish between Aristotle's method and the model-theoretic method. But it is worth taking first, because it could reveal corruptions in the text or confusions in Ibn Sīnā's mind. Also the requirement R3 is irrelevant to the logical content, as we noted earlier. But it is still worth keeping an eye on, because Ibn Sīnā seems to have tried to observe it, and this could have affected his success in meeting the other requirements.

We will take the requirements in the order R1 (= R1′), R2, R2′. But some preliminary work is necessary. All of these requirements are in terms of what sentences are verified or falsified by an interpretation, and we will see that even when Ibn Sīnā's text is clear, there may still be uncertainties about what proposition is being verified or falsified. Section 7 will set out the raw text of Ibn Sīnā's interpretations, with some bookkeeping to aid comparisons. Sections 8 and 9 will discuss how the raw text should be read. Then we consider requirement R1 in Section 10, R2 in Section 11, and requirement R2′ in Sections 12 and 13.

7 The interpretations set out

We list all the interpretations that Ibn Sīnā gives in his sterility proofs. Each sterility proof uses two interpretations; we write M for the one that Ibn Sīnā gives first, and N for the one that he gives second. Then for example the first and second interpretations in Hyp3 are Hyp3M and Hyp3N. Some of the entries in the tables are abbreviated to save space; the originals are in Appendix A.

All five premise-pairs in triplets 1–3 of Section 4 are parity variants of the premise-sequence

(13) $$(a, mt)(r, q),\ (i, mt)(q, p).$$

For requirements R1 and R2′ it will be convenient to adjust the letters in Ibn Sīnā's interpretations of these premise-pairs so as to bring them all to interpretations of the same premise-sequence (13). The following lemma allows this. If π is a parity switch on the letters in a set L, and I is an interpretation, we apply π to I by negating the reading assigned to any letter in L by I. We will do this by writing $\neg$ in front of the reading. This makes clear what the original readings were for r and p before the switch; we will need to know this when we check requirement R2.

Lemma 3 *Let the interpretation I be a model of the sentence ϕ, and π a parity switch. Then πI is a model of $\pi\phi$.*

Proof. Intuitively clear; formal details are in [16]. □

interp	r	q	p
Hyp3M	Zayd is walking.	Zayd is changing place.	Zayd abstains from walking.
Hyp3N	It is musk.	It is black.	It is scented.
Hyp12M	It walks.	It performs an intention.	It moves.
Hyp12N	It walks.	It performs an intention.	It rests.
Hyp52M	This is even.	This is a number.	This is a power of 2.
Hyp52N	This is even.	This is a number.	This is odd-of-odd.
Hyp59M	¬ This is a number.	¬ This is even.	This is a white colour.
Hyp59N	¬ This is a number.	¬ This is even.	This is odd.
Hyp63M	¬ It is not a vacuum.	¬ It is even.	¬ It is odd.
Hyp63N	¬ It splits in halves.	¬ It is even.	¬ It is odd.

Triplets 4 to 6 consist of premises-pairs that are parity variants of the premise-pair

(14) $$(a, mt)(r, q),\ (a, mt)(\overline{q}, p).$$

Here are the relevant interpretations, adjusted to the premise-pair (14):

Hyp7M	This is even.	This is a number.	¬ This is a multiplicity splitting in halves.
Hyp7N	This is even.	This is a number.	¬ This is a multiplicity not splitting in halves.
Hyp10M	Such-and-such is a human.	It is an animal.	¬ It is a flier.
Hyp10N	Such-and-such is a human.	It is an animal.	¬ It is rational.
Hyp16M	This is an accident.	This has a carrier.	This is a substance.
Hyp16N	This is an accident.	This has a carrier.	Dimensions are finite.
Hyp19M	This is a human.	¬ This is a stone.	This is a mineral.
Hyp19N	This is a human.	¬ This is a stone.	This is a body.
Hyp22M	This is an accident.	¬ This is a substance.	¬ This is in a subject.
Hyp22N	This is an accident.	¬ This is a substance.	¬ There is an infinite.
Hyp47M	¬ This is a vacuum.	¬ This is even.	This splits in halves.
Hyp47N	¬ This is a power of 2.	¬ This is even.	This splits in halves.
Hyp53M	This is not rational.	¬ It is a human.	It is an animal.
Hyp53N	This is a vacuum.	¬ It is a human.	It is an animal.
Hyp60M	¬ The human is not a body.	It is mobile.	It is a body.
Hyp60N	¬ The human is not a body.	It is mobile.	It is a vacuum.

Triplet 7 carries premise-pairs that are parity variants of:

(15) $$(a, mt)(q, r), (a, mt)(q, p)$$

There are two adjusted interpretation-pairs:

Hyp29M	This splits in halves.	This is even.	This is a number.
Hyp29N	This splits in halves.	This is even.	There is a vacuum.
Hyp36M	Zayd is in water.	Zayd is drowning.	$\neg$ Zayd is flying.
Hyp36N	Zayd is in water.	Zayd is drowning.	$\neg$ There is a vacuum.

Finally for triplet 8 we use the premise-pair

(16) $$(a, mt)(r, q),\ (a, mt)(p, q).$$

The two interpretations, which need no adjustment, are:

Hyp67M	It is moving.	It is a substance.	It is at rest.
Hyp67N	It is moving.	It is a substance.	It is changing place.

8 Refinement of interpretations

A glance through the interpretations listed in Section 7 suggests strongly that these interpretations rely on some general truths of the form 'Every A is a B'. For example everybody who walks moves, every power of 2 is an even number, every human is an animal, every stone is a mineral.

But a closer look shows that not all the suggested general truths are cut and dry. For example does a person who walks always do so intentionally? (Sleep-walkers? people being frogmarched?) Is every moving thing a body and a substance? (Hallucinations and visual illusions?) Is everything that flies an animal? (Kites? meteorites?)

Ibn Sīnā was well aware of this point. In his ^{c}*Ibāra* [21] 100.9–109.2 he cites as dubious:

(17)
Every man is able to impregnate women. (Eunuchs?)
Every bird lays eggs. (Bats?)
Every ship floats. (Toy or artistic ships made of stone?)
Every human is rational. (Dead humans?)
Every human is capable of laughing. (New-born infants?)

Ibn Sīnā's view is that the truth value of the general statement depends on how the speaker intends the subject term. Most of these sentences can be made true by choosing a narrow meaning of the subject term so as to exclude exceptions. Thus on a narrow reading of 'bird', bats are not birds. On a narrow reading of 'man', eunuchs are not men. We will describe this way of rescuing general truths as 'restrictive definition'. (Ebbesen [6] reports that the problem whether dead humans are humans became a popular talking point in late 13th century Scholastic logic.)

There are some other dubious general truths that will not be helped by restrictive definition. These are general statements with a negative subject term and an affirmative predicate term. An example may appear in Hyp16M, which is not a model of the premises unless

(18) Everything that is not an accident is a substance.

On Ibn Sīnā's understanding of accidents and substances, is this claim true? For example he says in several places that possibilities are not substances (cf. McGinnis [26] p. 184); but I could find no place where he says that every possibility is an accident. If it was an accident it would have to be an accident in a substrate, but what substrate could serve this purpose?

Any restrictive definition of 'accident' or 'substance' will make the claim harder to justify. What is needed is a different kind of device—not a retouching of the terms, but a restriction of the universe of discourse. If the ten categories between them exhaust all the kinds of existent, then we can rescue the claim (18) by restricting our discussion to existents; then mere possibles are no longer counted in 'everything'.

A discussion at *Qiyās* 94.5–7 may be relevant here. Ibn Sīnā observes that when we say 'Some B is a C', it will also be the case that some non-B is a non-C, because there always is such a thing, 'either existent or nonexistent'. (He says *lazima*; I take it this is a pseudoconclusion, not a logical consequence.) Of course there will be some existent thing that is neither a horse nor a stone, and some existent thing that is neither a moment of time nor a power of 2. But Ibn Sīnā adds 'or nonexistent', evidently because he thinks there might be counterexamples to the general pattern. Could 'neither an accident nor a substance' be such a counterexample? See Asadollah Fallahi [7] for further discussion of this obscure passage.

In fact the problem illustrated by (18) may be much more pervasive than just the premises of Hyp16. For every sterility proof by the model-theoretic method, at least one of the interpretations must falsify the sentence $(i, mt)(\overline{r}, \overline{p})$, in other words, it must verify the sentence $(a, mt)(\overline{r}, p)$. Thus for Hyp12 we need at least one of the following propositions to be true:

(19) Everything that doesn't walk moves.
Everything that doesn't move rests.

The best hope for making at least one of these true is to take the second with the universe of discourse restricted to physical bodies.

Does Ibn Sīnā ever refer to restrictions of the universe of discourse in interpretations? The answer is unclear. In Hyp60 he write a clause 'The human is not a body'. If 'the human' plays the same role as 'this' and 'it' (to be discussed in the next section), then 'human' serves to restrict the application of the term to humans, and this is in effect a restriction of the universe. But unfortunately Hyp60 is one of the items where we have strongest evidence of corruption or confusion, and we can't say that restricting the universe to humans will help for the sterility proof. In any case the clause could simply be shorthand for the self-contained sentence 'Humans are not bodies'.

If Ibn Sīnā was able to use animate pronouns like 'he' and 'she', then a use of them would indicate that the universe was restricted to humans, or at least to animals. But Arabic pronouns distinguish only number and grammatical gender, not animate versus inanimate. Accordingly I have used 'it' rather than 'he' or 'she' for these pronouns in the translations. Ibn Sīnā does use a distinctive language for quantifying over times, but this is hardly a recipe for restrictions of the universe more generally.

Nevertheless the idea of a restricted universe forces itself on anybody working on syllogisms with negative terms. When De Morgan in 1846, in his first paper on syllogisms [5] p. 2, launched a programme for taking negative terms seriously, his first step was to introduce the notion of the 'universe of a proposition'. This universe turns out to be the class of individuals quantified over by the quantifier of the proposition. So it's almost inevitable that Ibn Sīnā did in fact work with restricted universes in his treatment of PL3, even if he had no terminology for saying or thinking so.

These two devices, restrictive definition and universe of discourse, are both essential tools for making interpretations fit for purpose in sterility proofs. We will refer to them together as 'refinements' of interpretations.

9 What kind of quantification?

Even with the help of refinements, we still have a major problem to solve in reading Ibn Sīnā's interpretations. This is the large number of explicit or implied pronouns in his clauses. Thus in the first interpretation of Hyp52 the clauses are 'This is even', 'It is a number', 'It is a power of 2'. How are we to understand 'this' and 'it'? (Presumably they are meant to be coreferential.) Until these pronouns are given some anchorage in the world, the sentences don't say anything and can't strictly be considered either true or false.

A priori there are two possible solutions. One is that the anchorage can be provided by a further refinement of the interpretation, so as to make explicit what individual is referred to by 'this' and 'it'. The other is that the pronouns are read as variables of quantification, so that when the clauses are put as clauses of an AM sentence, the time quantifier of the sentence captures them and quantifies them. On this second view, the notion of 'times' has to be revised so as to include the assignment of references to pronouns. I will refer to the first solution as 'supplying references', and to the second as 'quantification over assignments'.

There is a passage at *Qiyās* 264.1–10 that must surely refer to this issue. Ibn Sīnā discusses whether the two following sentences are equivalent:

(20) Whenever this is a human it is an animal.
Every human is an animal.

They are not equivalent, he says, because there is a conflict in the first sentence between having a quantifier 'whenever' and having a demonstrative pronoun 'this' which refers to a particular individual. The second sentence has no demonstrative pronoun, so this conflict never arises in it.

The passage may need closer analysis, but my understanding of it is that Ibn Sīnā is not claiming that the first sentence can't be read as saying the same as the second. He is claiming that the first sentence has two readings, only one of which matches the second sentence. Reading One takes 'this' as referring to an individual who must be identified by the context. Reading Two, which is the one matching the second sentence, takes 'this' as being captured by the quantifier 'whenever'. Incidentally this is one of the few places where Ibn Sīnā does describe 'whenever' (*kullamā*) as a quantifier (*ḥaṣr*); normally he reserves the expression for determiners like 'all' and 'some'.

So Ibn Sīnā recognises both solutions. The solution of supplying references is the same as Reading One. This solution may appeal to some modern readers because it fits with standard modern practice. An interpretation or structure is meant to supply whatever information is missing in order to give truth values to the sentences of the relevant language. In this case the missing information is the reference of referential terms—think of them as individual constant symbols. But modern practice is not a good reason to read the idea back into Ibn Sīnā's text. In fact this solution leads to some awkwardnesses.

For example, finding negative pseudoconclusions for Hyp16 under this solution involves determining whether it is true that

(21) Whenever it is an accident, dimensions are not finite.

We can make this true by choosing 'it' to be something that isn't an accident. But can that really be what Ibn Sīnā had in mind? My own experience is that it is unmanageable to try to make sense of Ibn Sīnā's sterility proofs by this route. But readers are welcome to try it themselves.

Reading Two above corresponds to quantification over assignments. Pedantically speaking there is a difference between quantifying over individuals and quantifying over assignments of individuals to pronouns. But in practice they come to the same thing; the difference lies in how much we want to decorate the facts with semantic theory.

This second solution requires us to read 'whenever' as quantifying over more than just times. But on this point Ibn Sīnā meets us halfway, because he tells us in a number of places that when he speaks of 'times', he may mean situations rather than times, and they may be possible rather than actual. We see this in the ways that he qualifies or paraphrases quantifications over time. Thus:

(22)
Qiyās 41.14 'a condition equivalent to a time' (*ka-ḥukm al-waqt*)
Qiyās 367.1 'every posit (*waḍ*^{c}) of the antecedent'
Qiyās 379.6 'If there is some time or circumstance (*hāl*)'
Qiyās 383.9 'under some time and condition (*sharṭ*)'.

We also see it in examples that he gives of 'times', as for example

(23)
Qiyās 142.9f 'true at some time that every animal is a horse'
Qiyās 290.11f 'at a certain time when all fire is moving in the same direction'

(Cf. similar examples at *Qiyās* 30.11, 84.3f, 132.15f, 133.2, 134.11f, 138.10 etc.) Commenting on this issue, Movahed [27] p. 14 points to *Qiyās* 265.1–5; here Ibn Sīnā says explicitly that the quantifier 'whenever' (*kullamā*) quantifies not just over occasions (*mirār*) but also over circumstances (*ḥāl*) and conditions (*sharṭ*). It seems from all these and similar passages that Ibn Sīnā allows 'whenever' to be read as a quantification over kinds of situation in which the antecedent is true.

This broadening is a help, but by itself it is not enough to get us to quantification over assignments or non-temporal individuals. Readers who want to pursue this point further might like to read the treatment of 'always' in the seminal paper [24] of David Lewis, and marvel at how close Lewis comes to the quantification of assignments solution above. Giolfo and Hodges [9] also note the close similarities between Lewis's paper and Ibn Sīnā on other aspects of hypothetical sentences.

Let us sum up what all this entails for making sense of the readings in Ibn Sīnā's interpretations. Consider four example sentences:

(24) Zayd is in water. (From Hyp36)

(25) This is a power of 2. (From Hyp47)

(26) There is an actual infinite. (From Hyp22)

(27) He is walking. (From Hyp12)

In (24), if Zayd is taken to be some identified individual, then the sentence is sure to be true at some times and false at others. So the quantifier $\forall\tau$ or $\exists\tau$ in an AM sentence containing (24) as a clause can be read straightforwardly as a quantifier over times. The sentence (25) by contrast has a hanging 'This' but nothing that depends on time. So when (25) is a clause of an AM sentence, the quantifier $\forall\tau$ or $\exists\tau$ of the AM sentence captures 'this' and we have a simple quantification over individuals. Sentence (26) has nothing that depends on time and no hanging pronoun; it can only be read as a self-contained true or false sentence (and we know Ibn Sīnā took it to be false). So when an AM sentence has (26) as a clause, the quantifier of the AM sentence is irrelevant to the clause.

There remains the case of (27). This sentence is time-dependent *and* has a hanging pronoun. So when it is a clause in an AM sentence, the quantifier of the sentence has to be read as quantifying both over times and over individuals, as for example

(28) For all times t and all individuals b, if b is walking at time t then ....

You can if you like regard the quantifier as quantifying over ordered pairs of a time and an individual. Ibn Sīnā himself recommends this approach in a related context (*Qiyās* ix.7, 476.2–17; this passage is translated in [18]).

So the general situation is that the time quantifiers of AM sentences must be read as double quantifiers quantifying over both times and individuals simultaneously, though in many cases one or other of these quantifications will be vacuous. If I is an interpretation and S a sentence given as a reading in I, then we define the 'extension' of S to be the set of all ordered pairs (t, x) (t a time and x an individual) which satisfy S (i.e. make it true). Two readings are 'coextensional' if they have the same extension. A reading is called 'empty' if its extension is the empty set (as for example with the sentence 'There is a vacuum'). A reading is called 'total' if its extension consists of all ordered pairs of a time and an individual. A reading is called 'extremal' if it is either empty or total. If an interpretation has a restricted universe, then the individuals are required to come from this universe in all the definitions of this paragraph.

Extremal readings appear several times in Ibn Sīnā's interpretations, and in Section 12 below we will see some mathematical results that explain why this has to be so.

10 Are the interpretations models of the premises?

One absolute sine qua non for a proof of sterility is that the interpretations given should be models of both the premises. This is requirement R1 or R1′; it is obvious to us and must have been obvious to Ibn Sīnā. It should also be easy to check, since Ibn Sīnā normally specifies an interpretation of a premise-pair by writing out the result of applying the interpretation to the premises (see Appendix A). So a sterility proof that violates this requirement is likely to contain something that Ibn Sīnā didn't intend. We will check whether the interpretations in Ibn Sīnā's text do meet the requirement.

For his sixteen sterility proofs Ibn Sīnā gives thirty-two interpretations, each of which is required to verify two premises. So there are sixty-four items to be checked. Our arrangement of the interpretations in Section 7 is meant to aid the checking. For example in the group from triplets 4–6, the second premise is given in (14) as $(a, mt)(\overline{q}, p)$. We check this for the interpretation Hyp10N by putting 'He is not an animal' for $\overline{q}$ and 'He is not rational' for p. (The two negations come from different places; the first is from the premise, the second is the $\neg$ and is the result of the parity switch that we introduced to bring the premise-pair to the form (14).) Putting these two sentences as the clauses of (a, mt) gives the sentence

(29) Everything that is not an animal is not rational.

In Ibn Sīnā's view only humans are rational, and all humans are animals, so the interpretation verifies this premise.

At present we can only check provisionally that the interpretations are models of the premises. This is because we may need to introduce refinements of the interpretations in order to meet requirement R2 or R2′. The main point to watch here is that an interpretation might cease to verify an existential premise if we restrict the universe of the interpretation to exclude some individuals.

Most of the sixty-four verifications are straightforward. We note those where something is clearly wrong. More serious failures:

Hyp29N As it stands, the second premise says that if this is even then there exists a vacuum. Since Ibn Sīnā himself says that the reading has to be 'existence of a vacuum' in order to get a negative pseudoconclusion, the most obvious correction is blocked.

Hyp47M The first premise requires that nothing is even. We have the combination of an (e, mn) sentence and an extremal reading, both of which can confuse. The simplest remedy is to replace 'This is a vacuum' by 'This is not a vacuum'.

Hyp60 Ibn Sīnā states the second premise in the wrong form for his schedule, and as a result gives contradictory information about the reading of q. There is no obvious emendation, so we probably have to abandon this case as corrupt.

We also note some items where a refinement or a fine tuning of the definitions is needed:

Hyp3M The second premise requires that one can move while abstaining from walking.

Hyp10M The second premise requires that only animals fly.

Hyp12 For the first premise, walking has to be restricted to intentional walking.

Hyp16M The second premise requires that anything that is not an accident is a substance.

Hyp67 The first premise in both cases requires that anything that moves is a substance.

In sum, requirement R1 has discovered just three items where a premise is clearly false, and in one of these items there is a plausible emendation that saves the truth of the premise. Curiously both the remaining items refer to the existence of vacuums.

11 Do we have pseudoconclusions?

We turn to test the requirement R2 of pseudoconclusions, which is the central plank of Aristotle's method for proving sterility. This test requires us to use the same readings of r and p as in Ibn Sīnā's text; so in the listings of Section 7 we read only the English text below r and p, and ignore the $\neg$.

The requirement, for a given premise-pair, is that one of the two given interpretations verifies $(a, mt)(r, p)$ and the other verifies $(a, mt)(r, \overline{p})$. The model-theoretic method will also require that each of these sentences is verified by one or other of the two interpretations, though not necessarily the same one. So if we find Ibn Sīnā giving two interpretations which both falsify $(a, mt)(r, p)$, for example if both verify $(i, mt)(r, p)$ and $(i, mt)(r, \overline{p})$, then this spells trouble both for pseudoconclusions and for the model-theoretic method.

For example Hyp3N verifies $(a, mt)(r, p)$, since it is true that all musk is scented; and Hyp2M verifies $(a, mt)(r, \overline{p})$ since nobody who is walking is abstaining from walking. Most of Ibn Sīnā's interpretation pairs work equally smoothly. We will call an interpretation 'affirmative' if it verifies $(a, mt)(r, p)$ and 'negative' if it verifies $(a, mt)(r, \overline{p})$.

The problem cases are as follows.

- In Hyp36 both interpretations are negative.
- Hyp47N is negative. Hyp47M is both affirmative and negative, since 'This is a vacuum' is an empty reading. But the repair suggested in the previous section knocks out the affirmative pseudoconclusion.
- Hyp52N is negative, but neither interpretation is affirmative.
- Hyp53M is neither affirmative nor negative, at least without restricting the universe. But Hyp53N is both affirmative and negative, since 'This is a vacuum' is an empty reading.
- Hyp59M is negative, since no number is a white colour. But neither interpretation is affirmative, unless for example we restrict the universe of J to odd numbers.
- In Hyp60 both interpretations are negative.
- Hyp63N is negative. But the only way to make Hyp63M affirmative is to restrict the universe to odd numbers, and this falsifies the second premise.

Of these, Hyp36, Hyp47, Hyp53, Hyp60 and Hyp63 all involve vacuums. In fact the only other interpretation-pair involving vacuums is Hyp29, which was marked up as a problem in the previous section. A pattern seems to be emerging.

12 Complementary interpretations

The model-theoretic method for proving nonentailment was described in Section 6 above. It forms a method for proving sterility in PL3 when we use it to show that none of the eight candidate conclusions is entailed by the premises. If Ibn Sīnā is using this method, then he is using it in a constrained form where just two interpretations are given, and each of the eight candidate conclusions is ruled out by being falsified by one or other of the two interpretations.

By an 'rp-sentence' we will mean an AM sentence whose first term letter is r and whose second term letter is p. There are eight rp-sentences, and they are exactly the eight candidate conclusions for the premise-pairs considered from Section 6 onwards. They were listed at (9) above.

Fact 4 *Let I and J be interpretations (which assign sentences to the letters r and p and possibly other letters). Then the following are equivalent:*

(a) J falsifies exactly the rp-sentences that I verifies.

(b) Every rp-sentence is verified by at least one of I and J.

(c) Every rp-sentence is falsified by at least one of I and J.

(d) The existential rp-sentences verified by I are exactly those falsified by J.

Proof. The rp-sentences fall into four pairs consisting of an existential sentence and its universal contradictory negation. Each of I and J verifies one sentence in each contradictory pair and falsifies the other. So each of I and J verifies exactly four rp-sentences and falsifies exactly four, and hence (b) and (c) both imply (a). The remaining implications are straightforward. □

When the equivalent conditions of the lemma hold, we say that I and J are a 'complementary pair' of interpretations.

Fact 5 *For a pair of interpretations I and J to form a proof of the sterility of a premise-pair ϕ, ψ, it is necessary and sufficient that*

(a) Both I and J are models of ϕ and ψ, and

(b) I and J are a complementary pair.

Proof. Clear from the definitions and Fact 4(c). □

Since we have already considered which of Ibn Sīnā's interpretations are models of the premises, it remains to check which of his interpretation-pairs are, or can be refined to be, complementary pairs. (But remember that after refining interpretations we need to check that they are still models of the premises.) To check directly that a pair of interpretations is complementary, we must check for each of the eight rp-sentences that it is falsified by at least one of the two interpretations. This check is not arduous, but it is confusing and error-prone. Fortunately there are some mathematical facts about complementary pairs that lessen the risks.

The first fact that we note has the merit that if I and J violate any of its clauses (a)–(d), then so will any pair of interpretations got by restricting the universes of I or J. So a failure of any of these clauses is serious.

Fact 6 (Extremal Case Test) *Suppose I, J are a complementary pair of interpretations. Then:*

(a) If p is total in J then I verifies both $(i, mt)(r, \overline{p})$ and $(i, mt)(\overline{r}, \overline{p})$.

(b) If p is empty in J then I verifies both $(i, mt)(r, p)$ and $(i, mt)(\overline{r}, p)$.

(c) If r is total in J then I verifies both $(i, mt)(\overline{r}, p)$ and $(i, mt)(\overline{r}, \overline{p})$.

(d) If r is empty in J then I verifies both $(i, mt)(r, p)$ and $(i, mt)(r, \overline{p})$.

Proof. (a) If p is total in J, then J falsifies both $(i, mt)(r, \overline{p})$ and $(i, mt)(\overline{r}, \overline{p})$, so I must verify both sentences. The same argument gives (b)–(d). □

The next two lemmas will give us useful hints on where to look for complementary pairs.

Fact 7 *Suppose I, J are a complementary pair of interpretations, and in I the extension of r is disjoint from that of p and both are nonempty. Then in J, r and p are coextensional and nonempty.*

Proof. The supposition implies that I verifies all of $(i, mt)(r, \overline{p})$, $(i, mt)(\overline{r}, p)$ and $(a, mt)(r, \overline{p})$, so J must falsify all of these. □

Fact 8 *Suppose I, J are a complementary pair of interpretations, and in I, r is empty and p is not empty. Then in J, the extension of p is a proper nonempty subset of the extension of r. Moreover either r is total in J or p is total in I, but not both.*

Proof. The supposition implies that I verifies $(i, mt)(\overline{r}, p)$, $(a, mt)(r, p)$ and $(a, mt)(r, \overline{p})$. So J falsifies all of these, and hence the extension of p in J is a proper nonempty subset of that of r. Under these conditions, r is total in J if and only if J falsifies $(i, mt)(\overline{r}, \overline{p})$, and p is total in I if and only if I falsifies $(i, mt)(\overline{r}, \overline{p})$. □

The next two lemmas give reasons why we should expect to see a number of interpretation-pairs that make either r or p extremal.

Write $\mathcal{E}$ for the set of existential rp-sentences. If I and J are a complementary pair, then they partition the set of existential rp-sentences into two disjoint sets (not necessarily both nonempty), namely those verified by I and those verified by J. This partition will be described as the 'partition induced by I, J'. There are eight possible partitions of $\mathcal{E}$, one containing the empty set, four containing singleton sets and three partitions into two sets of two sentences. Every such partition is induced by some complementary pair of interpretations of r and p, but not all of these complementary pairs can be expanded to models of a particular premise-pair. Given a premise-pair ϕ, ψ and a partition P of $\mathcal{E}$, we say that P is 'admissible for ϕ, ψ' if there is a complementary pair of models of ϕ, ψ which induces P.

Fact 9 *There is a unique partition of $\mathcal{E}$ which is induced by interpretation-pairs where neither interpretation makes either r or p extremal.*

Proof. Let I, J be a complementary pair, neither of which makes r or p extremal, and let P be the induced partition. Since r is not total in either I or J, $(i, mt)(r, p)$ and $(i, mt)(r, \overline{p})$ are in different partition sets of P. By the same argument with p, $(i, mt)(r, p)$ and $(i, mt)(\overline{r}, p)$ are in different partition sets of P. Using these results and the corresponding results for 'empty' in place of 'total', we find that P has the two partition sets $\{(i, mt)(r, p), (i, mt)(\overline{r}, \overline{p})\}$ and $\{(i, mt)(\overline{r}, p), (i, mt)(r, \overline{p})\}$. This determines P uniquely among the eight possible partitions. □

The interpretations given for Hyp7, Hyp10 and Hyp67 in the next section are all illustrations of Fact 9.

Fact 10 *In a complementary pair I, J, if r is extremal in I but not in J, then p is extremal in I. The same holds with r and p transposed.*

Proof. Suppose first that r is empty in I and p is not extremal in J. Then by Fact 8, r is extremal in J. The case where r is total is got by applying the same argument to $\overline{r}$. Then apply the same arguments with r and p transposed. □

Fact 10 indicates that the number of extremal readings given by a complementary pair of interpretations is never just 1. If Ibn Sīnā gives a pair of interpretations where this number appears to be 1, as for example in Hyp16, Hyp22, Hyp29, Hyp36, Hyp47, Hyp53, Hyp60 or Hyp63, we should expect to have to impose a restriction on a universe in order to get another instance of r or p being extremal.

The next facts apply specifically to sterile premise-pairs of one of the two types $\forall\exists$ and $\forall\forall$.

Fact 11 *Suppose ϕ, ψ is the sterile premise-pair $(a, mt)(r, q), (i, mt)(q, p)$ (i.e. (13)). Then:*

(a) The admissible partitions for ϕ, ψ are those in which the sentences $(i, mt)(r, p)$ and $(i, mt)(\overline{r}, p)$ are in different partition sets.

(b) In any complementary pair of models of the premises, one of the interpretations satisfies the conditions on I in either Fact 7 or Fact 8 above.

Proof. (a) If $(i, mt)(r, p)$ and $(i, mt)(\overline{r}, p)$ are in the same partition set, then in any complementary pair of models of the premises, one of the interpretations falsifies both of these sentences, making p empty. But this is impossible in a model of $(i, mt)(q, p)$.

(b) Consider the partition induced by the complementary pair. Suppose $(i, mt)(r, \overline{p})$ is in the same partition set as $(i, mt)(\overline{r}, p)$. Then the interpretation represented by this partition set satisfies the condtions on I in Fact 7. On the other hand if $(i, mt)(r, \overline{p})$ is in the other partition set, then the interpretation represented by that other partition set satisfies the conditions on I in Fact 8. □

It is shown in [16] that if we are in type $\forall\forall$, i.e. ϕ and ψ are both universal, then every partition of $\mathcal{E}$ is admissible for ϕ and ψ. But there is an important restriction on the possible sterility proofs for this type. The underlying reason for the restriction is that if we had two interpretations which proved the sterility of the premise-pair (15), and both interpretations made q nonempty, then the same pair of interpretations would serve to prove the invalidity of the mood *Darapti* in categorical logic.

Fact 12 *Suppose that ϕ, ψ is a sterile premise-pair consisting of two universal sentences, with shared term letter q, and suppose I, J are a complementary pair of models of ϕ and ψ.*

(a) If q is distributed in both ϕ and ψ, then q is empty in at least one of I and J.

(b) If q is undistributed in both ϕ and ψ, then q is total in at least one of I and J.

(c) q is extremal in at least one of I and J.

Proof. (a) and (b) are proved in [16]. Then (c) follows by Theorem 2 above. □

13 Are all candidates ruled out?

Armed with the mathematical information in the previous section, we will check each one of Ibn Sīnā's sterility proofs to see whether it can be read as a valid proof of sterility. For these it will be easiest to work with the revised forms given in Section 7, i.e. taking account of $\neg$ in the readings.

We begin with the $\forall\exists$ group. Thanks to the reduction of this group to the premise-pair (14), Fact 11(b) applies and tells us that we must look for an interpretation-pair either as in Fact 7 or as in Fact 8. This gives us a strategy for finding sterility proofs, as follows.

Given interpretations I and J, we first inspect I and J to see whether either of them can be read as making r disjoint from p, and both of them nonempty. Suppose I does this. Then by Fact 7 we know that we need to have r and p coextensional and nonempty in J. With luck a restriction of the universe of J will achieve this. If it can be done, we then consider the sentence $(i, mt)(\overline{r}, \overline{p})$. Exactly one of I and J must verify this. We aim to restrict the universe of exactly one of I and J to the union of the extensions of r and p. If all this can be done, we have made I and J into a complementary pair. If one of the two premises was existential, then we must still check whether the restrictions imposed on the universes of I and J are not so strong as to falsity the existential premise.

If on the other hand neither of I and J can be read as making r disjoint from p, then we know that we need one of I and J to make r empty and p not empty. Then we continue as in Fact 8.

This strategy finds the following refinements of Ibn Sīnā's interpretation-pairs for the first four premise-pairs of type $\forall\exists$.

Hyp3		
M	universe	all times
	r	time when Zayd walks
	q	time when Zayd moves
	p	time when Zayd deliberately doesn't walk
N	universe	all musk
	r	musk
	q	black things
	p	things that smell pleasant

Hyp12		
M	universe	$\{(t,x) : x$ walks at time $t\}$
	r	x walks at time t
	q	x acts deliberately at time t
	p	x moves (= walks) at time t
N	universe	times $\times$ individuals
	r	x walks at time t
	q	x acts deliberately at time t
	p	x rests (= doesn't walk) at time t

Hyp52		
M	universe	powers of 2
	r	even numbers
	q	numbers
	p	powers of 2
N	universe	natural numbers
	r	even numbers
	q	numbers
	p	odd-of-odd numbers

Hyp59		
M	universe	white colours
	r	non-numbers
	q	things not even numbers
	p	white colours
N	universe	natural numbers
	r	non-numbers
	q	things not even numbers
	p	odd numbers

The remaining premise-pair of this type is Hyp63. If we take 'doesn't split in halves' and 'is not odd' to be disjoint, by restricting the universe of Hyp63N to numbers, we have to make 'is not odd' and 'is a vacuum' coextensional in Hyp63M, so that both are empty in Hyp63M. This will prevent Hyp63M from being a model of the second premise. So we try the approach by Fact 8 instead, taking r empty and p nonempty in Hyp63M. This leads to success with the following refinement:

Hyp63		
M	universe	everything except even numbers
	r	it is a vacuum
	q	it is not even
	p	it is not odd
N	universe	everything except even numbers
	r	it doesn't split in halves
	q	it is not even
	p	it is not odd

For the $\forall\forall$ premise-pairs we no longer have Fact 11 to help us. But it will still be worthwhile to try the strategy suggested by Fact 7, since the partition used there is admissible for sterile $\forall\forall$ premise-pairs, and we have already seen that Ibn Sīnā has used sterility proofs based on this partition.

In four cases this approach is rewarded with success:

Hyp7		
M	universe	natural numbers
	r	even numbers
	q	natural numbers
	p	things not even numbers
N	universe	natural numbers
	r	even numbers
	q	natural numbers
	p	things not odd numbers

Hyp10		
M	universe	non-swimming non-crawling animals
	r	humans
	q	animals
	p	non-fliers
N	universe	animals
	r	humans
	q	animals
	p	non-humans (= non-rational beings).

Hyp19		
M	universe	bodies
	r	humans
	q	non-stones
	p	minerals
N	universe	humans
	r	humans
	q	non-stones
	p	total (= dimensions are finite)

Hyp67		
M	universe	times $\times$ substances
	r	x is moving at time t
	q	x is a substance at time t
	p	x is at rest (= not moving) at time t
N	universe	times $\times$ substances
	r	x is moving at time t
	q	x is a substance at time t
	p	x is moving (= changing place) at time t.

Hyp53 looks difficult at first. But since q is undistributed at both occurrences in the premise-pair, Fact 12 alerts us that at least one of the interpretations must make q total, i.e. must exclude humans from the universe. That one move allows the rest to fall into place:

Hyp53		
M	universe	non-humans
	r	this is non-rational
	q	it is non-human
	p	it is an animal
N	universe	non-humans
	r	this is a vacuum
	q	it is a non-human
	p	it is an animal

The case of Hyp29 is already in trouble, because we saw in Section 10 that Hyp29N is not a model of the second premise. That is, unless we take the drastic step of making q empty by restricting the universe of Hyp29N to odd numbers. But Fact 12 has prepared us for this by warning that q has to be empty in at least one of the interpretations. The problem then is to make one of the interpretations verify $(i, mt)(r, \overline{p})$, since Ibn Sīnā apparently takes everything that is divisible into two equals to be an even number. This could be solved by replacing 'divisible into two equals' by 'divisible into three equals', in which case Hyp29N verifies $(i, mt)(r, \overline{p})$. This is certainly too much of a leap for us to have any confidence that Ibn Sīnā intended it. But it does show that his proof is repairable as follows:

Hyp29		
M	universe	odd numbers
	r	*This splits in thirds*
	q	this is even
	p	this is a number
N	universe	odd numbers
	r	*This splits in thirds*
	q	this is even
	p	there is a vacuum

That leaves five sterility proofs not validated, namely Hyp16, Hyp22, Hyp36, Hyp 47 and Hyp60. It happens that these are precisely those interpretation-pairs which fail the Extreme Case Test, Fact 6. Thus Hyp16N makes p total, so that Hyp16M must verify that some non-accident is a non-substance, contradicting the second premise. Likewise Hyp22N makes p total, so Hyp22M must verify that something in a substrate is not an accident. Likewise Hyp36N makes p total, and hence Hyp36M must verify that at some time Zayd is both flying and in the water. Then Hyp47 makes r total, so it must verify that some power of 2 fails to split in halves. Hyp60 makes p empty, so it must verify that some body is not a body.

This is an agreeably sharp result: ten of Ibn Sīnā's sterility proofs work, an eleventh almost works, and the remaining five all fail a test that it is hard to recover from. Explaining what mental state can have led Ibn Sīnā to these offerings is less straightforward. We turn to that question.

14 Verdicts

The following table lists our results on success and failure of Ibn Sīnā's sterility proofs under the three heads: Models, Pseudoconclusions, Complementary Pairs. We count a refinement as a success if it does the required mathematical job. That leaves open the objection that even if the refinement works, Ibn Sīnā would hardly have thought of it, and if by good chance he had thought of it he would surely have told us. I can't refute that objection. I would say only that what Ibn Sīnā actually does is more reliable guide to his understanding than what he says he is doing, given the obscure nature of many of his explanations. If the evidence of his arguments tends to suggest that he *did* think of ideas like restricting the universe, and that he *did* go ahead with these ideas without explaining them to us, then we should be prepared to take that evidence seriously.

	Models	Pseudoconclusions	Complementary Pairs	type
Hyp3	Yes	Yes	Yes	∀∃
Hyp7	Yes	Yes	Yes	∀∀
Hyp10	Yes	Yes	Yes	∀∀
Hyp12	Yes	Yes	Yes	∀∃
Hyp16	?	Yes	No	∀∀
Hyp19	Yes	Yes	Yes	∀∀
Hyp22	Yes	Yes	No	∀∀
Hyp29	No	Yes	Repairable	∀∀
Hyp36	Yes	No	No	∀∀
Hyp47	Repairable	Not if repaired	No	∀∀
Hyp52	Yes	No	Yes	∀∃
Hyp53	Yes	No	Yes	∀∀
Hyp59	Yes	No	Yes	∀∃
Hyp60	No	No	No	∀∀
Hyp63	Yes	No	Yes	∀∃
Hyp67	Yes	Yes	Yes	∀∀

The first point to note in this table is the large number of No's. In the columns for Models and Pseudoconclusions it is reasonably clear what Ibn Sīnā is aiming to do, so each of the No's in these columns indicates that there is a logical mistake somewhere along the line. By contrast a No in the Complementary Pair column might indicate a logical mistake by Ibn Sīnā or a copyist, but it could also count against the hypothesis that Ibn Sīnā is trying to use the model-theoretic procedure.

The number of mistakes is particularly striking when we set it against the situation with Ibn Sīnā's two-dimensional logic and the proof-theoretic part of PL3. In the case of the 2D logic, only one significant logical mistake is known, namely one one noted by Fak̲r al-Dīn al-Rāzī and Saloua Chatti [4] in connection with *wujūdī* sentences; and even that mistake may be a deliberate one made for a practical purpose. Other errors in the text have the look of occasional copying errors. In the case of the proof-theory of PL3, there are a very few places where we find what are probably either carelessnesses (such as the false claim to give a proof at Hyp71) or copying errors, but no evidence of any serious misunderstanding. So something has definitely gone wrong in the proofs of sterility.

Is there anything in the background of this part of Ibn Sīnā's work that might help to explain its unreliability? I think there is. The two-dimensional logic is already present in his early *Mukhtaṣar* [19], inchoate maybe and not clearly distinguished from the alethic modal logic. But Ibn Sīnā studied it in almost all of his surviving logical works (his Persian *Dāneshnāmeh* [23] is the only clear exception), and there is no sign of any change of direction in the logical content during the course of his career. So the two-dimensional logic is something he had worked over throughout his adult life. PL3 by contrast appears for the first time in *Qiyās*, written when he was in his mid forties. The sentence forms are mentioned in *Najāt* [20], and most of them are already in *Mukhtaṣar*, but virtually none of the proof theory in *Qiyās* vi.2 is found earlier than *Qiyās*, and there is absolutely no trace of the sterility proofs in Ibn Sīnā's earlier writings. So PL3 was a novel venture, and it would not be surprising to find that Ibn Sīnā had some parts of it under better control than others.

As to why the proof theory is in much better shape than the sterility proofs: this could be explained by a tendency in Ibn Sīnā's philosophy of science to concentrate on understanding regularities and rules, leaving exceptional cases unexplained. He quite possibly felt that sterility proofs wouldn't be truly scientific unless—against all probabilities—one could see them as applications of general laws. Ibn Sīnā wasn't in a position to assume, with Frege [8] p. 426, that 'this new realm [of nonentailment proofs] has its own specific, basic truths which are as essential to the proofs constructed in it as the axioms of geometry are to the proofs of geometry'.

It should be added that the practice of constructing sterility proofs is very confusing. Probably everybody makes mistakes; one has to check and then re-check. Unlike Ibn Sīnā, I had the advantage of some C++ programs for checking the various claims.

When we turn to the column on Complementary Pairs, the fact that there are ten sound sterility proofs recorded here is striking, even granting that the proofs need important details that Ibn Sīnā himself never mentioned. I think if we had only the ten cases marked with Yes in this column, without the other six, then it would be generally accepted that Ibn Sīnā had successfully created a model-theoretic machinery for proving nonentailment. Even the use of universes of interpretations would be credited to him. But the presence of the five 'No's in the column must put that verdict in doubt.

We noted that in all five cases of a straight 'No', Ibn Sīnā committed the same fatal oversight of failing the Extremal Case Test. So it may be that the main cause of the failures is a single misunderstanding that Ibn Sīnā made early on and never got out of his system. It shows up particularly with vacuums because vacuums are his favourite device for getting extremal clauses. If he had been fluent with model-theoretic methods he would have cleared up this misunderstanding; we deduce that he was not fluent with them.

The evidence from the table does broadly confirm hat Ibn Sīnā was trying to operate Aristotle's method of pseudoconclusions most of the time, even though he should have seen that the method is inappropriate for this logic. Could he perhaps have felt that he didn't have in his hands the justification for abandoning Aristotle's procedures and moving instead to an original new method? Perhaps he was neither fluent nor confident in this new territory.

Appendix A: Translations

(The translations below are taken from [17], where the whole of *Qiyās* vi.2 is translated.)

Hyp3, verbal form A113, *Qiyās* 305.12–306.2

When the difference-like premise is existential, [the premise-pair] is not productive. Terms to show this are firstly:

> Whenever Zayd is walking, he is changing place;
> and Zayd is changing place at some time other than when he is abstaining from walking.

And secondly:

> Whenever this is musk, i.e. without any other condition, then it is coloured black;
> and this thing is coloured black in some case other than when it has a pleasant smell.

/306/ In the first case the affirmative universal [sentence] is true, and in the second case the negative universal [sentence] is true.

Hyp7, verbal form A131, *Qiyās* 306.14–17

[We consider] the moods of this case, where only the difference-like premise is negative. Nothing is yielded by them. You can see this from the following matters.

> Whenever this is even, then it is a number;
> and other than when it is a number, it is never a multiplicity which is divisible into two equals.

This is one case, and the other case is that

> [It is never, other than when it is a number,] a multiplicity which is not divisible into two equals.

The one case makes an affirmative universal [sentence] true, and the other case makes a negative universal [sentence] true.

Hyp10, verbal form A211, *Qiyās* 307.5–9

[We consider] the moods of this case where the premise-pairs consist of two affirmative premises. Let them both be universal: 307.5

> Whenever H is Z, without any other condition, C is D;
> and always either C is D or A is not B.

This is not productive. The first example is:

> Whenever such-and-such is a human, it is an animal;
> and always either it is an animal, or it is not a flier.

The second case is:

> Either it is an animal or it is not rational.

Hyp12, verbal form A213, *Qiyās* 307.9–13

Likewise, when its difference-like premise is existential it mustn't have a conclusion. The first example is: 307.10

> Whenever it is walking it is exercising an intention.
> And it is sometimes the case that it is exercising an intention, other than when it is not moving.

Also

> It is sometimes the case that it is exercising an intention, other than when it is not at rest i.e. not intentionally at rest.

In fact one of these two matters yields the contrary of what the other yields.

Hyp16, verbal form A231, *Qiyās* 308.7–11

[We consider] the moods of this case where the difference-like premise is negative. None of these entail a conclusion. Terms [to show this] are [firstly]:

> Whenever this is an accident it has something that carries it (absolutely, without any condition);
> and other than when this has a carrier, it is never not a substance.

And [secondly]:

> Other than when [this] has a carrier, it is never the case that not every dimension is finite.

where no other corruptible condition is imposed on it. Incompatible [sen- 308.10
tences] follow from these terms.

Hyp19, verbal form A311, *Qiyās* 308.13–16

[We consider] the moods of this case where the premise-pair consists of two affirmative premises:

> Whenever H is Z then C is not D;
> and either C is not D or A is B.

[The premise-pair] is not productive. Matters [to show this] are [firstly]:

> Whenever this is a human, it is not a stone[a]; 308.15
> and either it is not a stone, or it is a mineral.

And [secondly]:

> Either it is not a stone, or it is a body.

[a] For *ᶜaraḍan* read *ḥajaran*.

Hyp22, A331–A333, *Qiyās* 309.1–7

/309/ [We consider] the moods of this case where the difference-like premise is negative. These [premise-pairs] are not productive. Let us give a single example of them:

> Whenever this is an accident, it is not a substance;
> and it is never the case, other than when this not a substance, that it is in a substrate;
> and it is never the case, other than when it is not[a] a substance, that some dimension is actually infinite.

It will not be hard for you to see, from [the cases considered] above, 309.5
that when the difference-like premises have both their clauses negative, the resulting premise-pairs behave in the same way as when the shared clause in the difference-like premise is negative and the other clause is affirmative.

[a] Add 'not' (as Shehaby).

Hyp29, verbal form B131, *Qiyās* 311.1–6

/311/ [We consider] the moods of this case where the difference-like premise is negative:

> Whenever H is Z, C is D;
> and it is never the case that H is Z, other than when A is B[a].

This is not productive. The reason is that when you say:

> Whenever this is even, it is divisible into two equals;
> and it is never the case that this even, other than when it is a number.

what is true is that

> Whenever this is divisible into two equals, it is a number.

And if you replace 'number' by 'existence of a vacuum', what is true is a 311.5
negative proposition.

[a] Reading ab for jd, with some mss.

Hyp36, verbal form B231, *Qiyās* 312.3–5

[We consider] the moods of this case where the difference-like premise is negative. These are not productive. Terms are:

> Whenever Zayd is drowning, Zayd is in the water;
> and it is not the case that Zayd is drowning other than when he is not flying;
> and it is not the case that Zayd is drowning other than when the 312.5
> vacuum doesn't exist.

Hyp43, verbal form B331, *Qiyās* 313.1–4

/313/ [We consider] the moods of this kind, where the difference-like premise is negative. This case is not productive. Term examples [can be taken] from the terms of similar cases,[a] but in place of the sentence 'he is drowning' put 'he is not drowning'.

Moods where their difference-like premises have both subclauses negative don't behave any differently; you should have no trouble seeing this.

[a] Delete 313.2 *wa-al-muttaṣilatu ḥaqīqiyyatun*. The author of these words has confused B231 with B131, a mistake easily made by a reader.

Hyp47, verbal form C121, *Qiyās* 313.15–314.2

[We consider] the moods of this kind where the difference-like premise is negative: 313.15

> It is never the case, other than when H is Z, that C is D;
> and whenever C is D, then A is B.

This is not productive. An instance of it with [material] terms is:

> /314/ It is never the case, other than when this thing is a vacuum, that it is even;
> and whenever it is even, then it is divisible into two equals.

Then instead of 'vacuum' put 'power of 2'.

Hyp52, verbal form C213, *Qiyās* 314.12–15

If the meet-like premise is existential, then it doesn't have to yield a conclusion. An instance of it with [material] terms is:

> Always either this is not even, or it is a number;
> and sometimes when it is a number, it is a power of 2.

Also

> When it is a number, it is odd times odd.

Hyp53, verbal form C221, *Qiyās* 314.16–18

[We consider] the moods of this kind where the difference-like premise is negative. These are not productive. [Material] terms are:

> It is never the case, other than when this is not not rational, that it is human.
> And whenever it is human, it is an animal.

Then put 'vacuum' in place of 'not rational' .

Hyp56, verbal form C233, *Qiyās* 315.5f

If the meet-like premise is existential, it is not productive. [Material] terms to show this are as in the case of two affirmative premises, taking into account that you are swapping an existential affirmative premise for an existential negative premise.

CHyp59, verbal form C313, *Qiyās* 315.11–14

If the meet-like premise is existential, [the premise-pair] is not productive. An instance of it with [material] terms is:

> Always either this described thing is a number, or else it is not even;
> and sometimes, when this is not even, it is a white colour.

Or with 'odd' for 'a white colour'.

Hyp60, verbal form C321, *Qiyās* 315.15–17

[We consider] the moods of this kind where the difference-like premise is negative. [The first one] is not productive. An instance of it with [mate- 315.15
rial] terms is:

> It is never the case, other than when the human is not a body, that it is not mobile;
> and whenever it is mobile, it is a body.[a]

Then in place of 'body' put 'vacuum'.

[a] Ibn Sīnā's schedule here requires a sentence of the form $(a, mt)(\overline{q}, r)$, but the meaning of q as the affirmative 'it is mobile' is given by the first premise. There is no obvious emendation.

Hyp63, verbal form C333, *Qiyās* 316.5–8

When the meet-like premise is existential, [the premise-pair] is not productive. [Material] terms [to show] that are as follows. One instance is: 316.5

> It is never the case that it is a vacuum, other than when it is not even;[a]
> and it is not the case that whenever[b] it is not even, it is odd.

Another instance is:

> It is never the case that it is not divisible into two equals, other than when it is not even[a];
> and it is not the case that whenever[c] it is not even it is odd.

[a] Ibn Sīnā seems to have forgotten his own schedule and used a negative sentence where an affirmative one is required. We will assume he meant 'It is always either not a vacuum or not even' and 'It is always either divisible into two equals or not even'.
[b] In 316.6, for *ḥukman* read *kullamā*. (Shehaby proposes *idhā*.)
[c] In 316.8, for *kullu mā* read *kullamā*.

Hyp67, verbal form D121, *Qiyās* 317.1–3 (Hyp67)

/317/ [We consider] the moods of this kind where the difference-like premise is negative. These are not productive. [Material] terms to show this [for the first example] are either:

> It is never the case that it is moving other than when it is a substance;
> and whenever[a] it is at rest it is a substance.

or

> Whenever it is changing place it is a substance.

[a] For *kullu mā* read *kullamā*, as Shehaby.

Hyp71, verbal form D211, *Qiyās* 317.14–16

[We consider] the moods of this kind [where the premise-pair consists] of two affirmative premises:

> Always either H is not Z or C is D;
> and whenever A is B, C is D.

The difference-like premise is brought to the following form:

> Whenever H is Z, C is not D;
> so it is never the case, if H is Z, that C is D.

Then the rest of the discussion is as you know.

Hyp72, verbal form D221, *Qiyās* 318.1–3

/318/ [We consider] the moods of this kind where the difference-like premise is negative:

> It is never the case, other than when H is Z, that C is D;
> and whenever C is D, A is B.

This also is not productive. [Material] terms [to prove this] are as in a similar case, but putting 'is not at rest' instead of 'moving' in the difference-like premise.

Appendix B: Definitions

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